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# Existence and Approximation of Traveling Wavefronts of a Diffusive Hematopoiesis Model with Time Delay

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**Resumen**

In this work, we consider a diffusive hematopoiesis model with discrete delay:

$$u_t(t,x) = \Delta u(t,x) + \frac{pu(t-\tau,x)}{1+au^q(t-\tau,x)} - du(t,x).$$

The above equation, without spatial dispersion, was proposed por Mackey-Glass in 1975. This model describes the dynamics of the blood cell production and $\tau$ is the time delay between the production of immature stem cells in bone marrow and their maturation for release in the circulating blood stream.. We investigate the existence of traveling wavefronts solutions connecting the two steady states of the model. We develop an alternative proof of the existence of such solutions, including the existence of traveling wavefronts moving at minimum speed. The proposed approach is based on the use technique of upper-lower solutions. Finally, through the iterative procedure developed in [1], we show numerical simulations that approximate the traveling wavefronts, thus confirming our theoretical results.

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## 1. Introduction

Recently there have been intensive studies on the existence of traveling waves to delayed reaction-diffusion equations arising from biological and physical applications, see, for example, the books Britton [12], Fife [13], Murray [14], Volpert et al [11]. Schaaf [2] is the pioneer work, where systematically studied is a scalar reaction-diffusion equation with a single discrete delay by using the phase-plane technique, the maximum principle for parabolic functional differential equations and general theory for ordinary functional differential equations. For delayed nonlinear reaction-diffusion equations with quasimonotonicity, nonlinearity, monotone-convergence scheme coupled with the standard upper-lower solution technique, developed by J. Wu and X. Zou [1], is widely used to establish the existence of monotone waves. See, for example [2-11,18].

In this work, traveling wavefront solutions including waves that move at minimum speed, will be investigated for the partial differential equation

$$u_t(t,x) = \Delta u(t,x) + \frac{pu(t-\tau,x)}{1+au^q(t-\tau,x)} - du(t,x), \tag{1}$$

where $x \in \mathbb{R}$, $t > 0$, $u \geq 0$, and all parameters are positive constants. The results obtained in [25] describe the behavior the oscillatory behavior of solutions about the positive equilibrium of (1) with Neumann boundary condition. Further, in [26] was investigated the existence of positive periodic solutions of equation (1) by using the Krasnosel Skii fixed point theorem. The above equation, without spatial dispersion, reduces to the following ordinary differential equation

$$u'(t) = \frac{pu(t-\tau)}{1+au^q(t-\tau)} - du(t), \tag{2}$$

this equation was first suggested by Mackey and Glass [17], among others, to model the concentration of cells in the circulating blood and $\tau$ is the time delay between the production of immature stem cells in bone marrow and their maturation for release in the circulating blood stream. this has been studied in [22-24]. In fact, biological backgrounds and detailed analysis have been presented by O. Arino, M.L. Hbid and E. Ait Dads [20]. A traveling wavefront solution to equation (1) is a special type of bounded positive continuous non-constant solution $u(t,x)$ having the form $u(t,x) = \phi(x+ct)$. The number $c > 0$ is called the wave speed of the propagation, and $\phi : \mathbb{R} \to \mathbb{R}$ is a $C^2$-smooth function called the wave profile and satisfying $\phi(-\infty) = 0$, $\phi(+\infty) = k$. The existence of the traveling wavefronts in (1) is equivalent to the presence of positive heteroclinic connections in an associated second order non-linear differential equation.

In [18], Z. Ling and Z. Lin investigated the existence of traveling wavefront solutions, for all $c > c_*$ (minimum speed), to equation (1) using the upper-lower solutions method developed in [1]. In this work we give an alternative proof of the existence of such solutions. Moreover, using the ideas of [19,21], we extend the result of Ling and Z. Lin by proving the existence of traveling waves moving at minimum speed and through the iterative procedure developed in [1], we show numerical simulations that approximate the traveling wavefronts.

Let us state now the main result of this work.

**Theorem 1.** There exists $c_* > 0$ such that for every $c \geq c_*$, the equation (1) has a positive monotone traveling wavefront $u(t,x) = \phi(x+ct)$, connecting 0 with $k = \left(\frac{p-d}{ad}\right)^{1/q}$, if one of the following conditions holds:

(a) $1 < \dfrac{p}{d} \leq \dfrac{q}{q-1}$ if $q > 1$;

(b) $1 < \dfrac{p}{d} \leq +\infty$ if $0 < q \leq 1$.

The organization of this work is as follows. In Section 2, we will introduce some notations, and present one of the main theorems from Wu and Zou given in [1] that will be employed in this paper. Section 3 and section 4 is devoted to establishing the existence of traveling wavefront solution of (1) for $c > c_*$ and using the ideas of [19,21], an alternative proof of the existence of traveling waves moving at minimum speed $c = c_*$ is given. In Section 5, through numerical simulations, we numerically approximate the solutions traveling wavefronts solutions of (1).

## 2. Preliminaries

In this section, we introduce some important results, which will be used in our analysis. As we know, Wu and Zou in [1] developed a quite general and applicable theory to coupled reaction-diffusion systems with delay. For convenience, here we only present a simple version of their result.

Consider a scalar reaction-diffusion with discrete delay:

$$u_t(t,x) = \Delta u(t,x) + f(u(t,x), u(t-\tau,x)), \tag{3}$$

where $t \geq 0$, $x \in \mathbb{R}$, $u \geq 0$ and $f$ is a continuous function. Substituting the wave profile $u(t,x) = \phi(x+ct)$, satisfying $\phi(-\infty) = 0$, $\phi(+\infty) = k$, into (1) and denoting $x+ct$ still by $z$, we get

$$c\phi'(z) = \phi''(z) + f_c(\phi_z), \quad z \in \mathbb{R}, \tag{4}$$

where $f_c : X_c = C([-c\tau, 0], \mathbb{R}) \longrightarrow \mathbb{R}$ is defined by

$$f_c(y_z) = f(y_z^c(0), y_z^c(-\tau)), \qquad y_z^c(s) = y_z(cs) = y(z+cs), \quad s \in [-\tau, 0].$$

We assume the following conditions on $f$.

**(F1)** There Exists $k > 0$ such that $f_c(\hat{0}) = f_c(\hat{k}) = 0$, and $f_c(\hat{u}) \neq 0$ for $0 < u < k$, where $\hat{u}$ denotes the constant function taking the values $k$ on $[-c\tau, 0]$, i.e., $\hat{u}(s) = u$, $s \in [-c\tau, 0]$

**(F1)** (Quasimonotonicity). There Exists $\beta \geq 0$ such that

$$f_c(\phi_z) - f_c(\psi_z) + \beta[\phi(z) - \psi(z)] \geq 0 \tag{5}$$

with $\phi_z, \psi_z \in X_c$ and $0 \leq \psi_z(s) \leq \phi_z(s) \leq k$, $s \in [-c\tau, 0]$.

We look for traveling wavefronts solutions of (1) in the following profile set:

$$\Gamma = \left\{ \phi \in C(\mathbb{R}, \mathbb{R}) \quad : \quad \begin{array}{l} \text{i)} \quad \phi \text{ is nondecreasing in } \mathbb{R} \\ \\ \text{ii)} \quad \phi(-\infty) = 0, \quad \phi(+\infty) = k. \end{array} \right\}.$$

Next we define upper and lower solutions for (4).

**Definition 1.** A continuous function $\rho \in C(\mathbb{R}, \mathbb{R})$ is called an upper solution of (4) if $\rho'$ and $\rho''$ exist almost everywhere in $\mathbb{R}$, and they are essentially bounded on $\mathbb{R}$ and if the following inequality holds:

$$\rho''(z) - c\rho'(z) + f_c(\rho_z) \leq 0, \quad z \in \mathbb{R}. \tag{6}$$

A lower solution of (4) is defined in a similar way by reversing the inequality in (6).

Now, we are in the position to state a scalar version of theorem 3.6 of [1].

**Theorem 2.** Assume that **(F1)** and **(F2)** holds. Suppose that (4) has an upper solution $\bar{\phi} \in \Gamma$ and a lower solution $\underline{\phi}$ (which is not necessarily in $\Gamma$) satisfying:

**(S 1.)** $\underline{\phi}(z) \not\equiv 0, \quad z \in \mathbb{R}$

**(S 1.)** $0 < \underline{\phi}(z) \leq \bar{\phi}(z) \leq k, \quad z \in \mathbb{R}$.

Then, (4) have a solution in $\Gamma$ . That is, (3) has a traveling wave front with speed $c$.

# 3. Existence of traveling wavefront solution: Case $c > c_*$

Substituting the wave profile $u(t,x) = \phi(x+ct)$ in to equation (1), we obtain the following second-order functional differential equation

$$\phi''(z) - c\phi'(z) + f_c(\phi_z) = 0, \quad z \in \mathbb{R}, \tag{7}$$

with

$$\phi(-\infty) = 0, \quad \phi(+\infty) = k, \tag{8}$$

where

$$k = \left(\frac{p-d}{ad}\right)^{1/q} \tag{9}$$

and

$$f_c(\phi_z) = \frac{p\phi_z(-cr)}{1 + a[\phi_z(-cr)]^q} - d\phi_z(0). \tag{10}$$

From here on we will denote the differential operator $L$ by:

$$L_\phi = \phi''(z) - c\phi'(z) + f_c(\phi_z), \quad z \in \mathbb{R}. \tag{11}$$

**lemma 1.** If $\frac{p}{d} > 1$, then there exists $k > 0$ such that $f_c(\hat{0}) = f_c(\hat{k}) = 0$, and $f_c(\hat{u}) \neq 0$ for $0 < u < k$, where $\hat{u}(s) = u$, $s \in [-c\tau, 0]$.

**Proof.** Clearly $k = \left(\frac{p-d}{ad}\right)^{1/q} > 0$, since $\frac{p}{d} > 1$. So by computing the stationary states of (7), we get the result.

Next we show that $f_c$ satisfies quasimonotonicity condition.

**lemma 2.** If (a) or (b) holds, then for all $\beta \geq d$, $f_c$ satisfies the quasimonotonicity condition.

**Proof.** Let $\phi_z, \psi_z \in X_c$ be, such that $0 \leq \psi_z(s) \leq \phi_z(s) \leq k$, $s \in [-c\tau, 0]$. Then

$$f_c(\phi_z) - f_c(\psi_z) = p\left(\frac{\phi_z(-c\tau)}{1 + a[\phi_z(-c\tau)]^q} - \frac{\psi_z(-c\tau)}{1 + a[\psi_z(-c\tau)]^q}\right) - d\left(\phi_z(0) - \psi_z(0)\right).$$

We consider the function $g(y) = \dfrac{py}{1 + ay^q}$. we notice that $g'(y) = \frac{p[1+ay^q(1-q)]}{(1+ay^q)^2} \geq 0$, for all $y \in \mathbb{R}$, since $0 < q \leq 1$, in the case that $q > 1$, $g'(y) \geq 0$, for all $y \in \left[0, \frac{1}{(aq-a)^{1/q}}\right]$ and $0 < k \leq \left(\frac{1}{(aq-a)^{1/q}}\right)$, because $\frac{p}{d} \leq \frac{q}{q-1}$. So the function $g$ is non-decreasig. Therefore

$$f_c(\phi_z) - f_c(\psi_z) + d[\phi(z) - \psi(z)] \geq 0.$$

Note that lemma 1 and lemma 2 prove the hypotheses (**F1**) and (**F2**) respectively on $f$. Linearizing the equation (7) around the equilibria 0 and $k = \left(\frac{p-d}{ad}\right)^{1/q}$, we obtain

$$\psi_0(\lambda, c) = \lambda^2 - c\lambda - d + pe^{-\lambda c\tau}. \tag{12}$$

and

$$\psi_1(\mu, c) = \mu^2 - c\mu - d + p_1 e^{-\mu c\tau} \tag{13}$$

where $p_1 = \frac{d}{p}[p - q(p-d)] > 0$, when (a) or (b) is satisfied.

We plan to investigate the solution of (7)-(8) by using upper-lower solutions . Hence, it is necessary to verify the following propositions.

**Proposition 1.** There exists $c_* > 0$ such that for $c > c_*$, the equation $\psi_0(\lambda, c) = 0$ has two positive real roots, $0 < \lambda_1 < \lambda_2$ and $\psi_0(\lambda, c) > 0$ for all $\lambda \in \mathbb{R} \setminus [\lambda_1, \lambda_2]$.

**Proof.** We notice that the function $\psi_0(\lambda, .)$ is concave up, since $\frac{\partial^2 \psi_0}{\partial \lambda^2} = 2 + pc^2\tau^2 e^{-\lambda c\tau} > 0$. We also have that $\psi_0(0, c) = p - d > 0$, $\psi_0(+\infty, c) = +\infty$, so if we choose $\lambda = \frac{c}{2}$ we have

$$\psi_0(\frac{c}{2}, c) = \left(\frac{c}{2}\right)^2 - c\left(\frac{c}{2}\right) - d + pe^{-\lambda \frac{c}{2}\tau} < p - d - \frac{c^2}{4},$$

thus, if $\psi_0(\frac{c}{2}, c) < 0$, then $c > 2\sqrt{p-d}$. Hence $c_* < 2\sqrt{p-d}$.

**Proposition 2.** For any $c > 0$, the equation $\psi_1(\mu, c) = 0$ has two real roots $\mu_1 < 0 < \mu_2$ and $\psi_0(\mu, c) > 0$ for all $\mu \in \mathbb{R} \setminus [\mu_1, \mu_2]$.

**Proof.** We notice that the function $\psi_1(\mu, .)$ is concave up, since $\frac{\partial^2 \psi_1}{\partial \mu^2} > 0$. We also have that $\psi_1(0, c) = -\frac{dq(p-d)}{p} < 0$ and $\psi_0(\pm\infty, c) = +\infty$, then we have the result.

In the remainder of this section, we construct for (7) an upper solution and a lower solution satisfying the conditions **(S1)** and **(S2)**.

**Proposition 3.** For any $c > c_*$ and $\eta \geq \eta_0 > 0$. The function

$$\bar{\phi}(z) = \begin{cases} \dfrac{k(\eta - \mu_1)}{\lambda_1 - \mu_1 + \eta} e^{z\lambda_1} & \text{si } z < 0 \\ k - \dfrac{\lambda_1 k}{\lambda_1 - \mu_1 + \eta} e^{z(\mu_1 - \eta)} & \text{si } z \geq 0, \end{cases} \tag{14}$$

where $\eta_0 = \dfrac{2\mu_1 - c + \sqrt{(c - 2\mu_1)^2 + 4p_1 e^{-\mu_1 c\tau}}}{2}$, is upper solution of (7).

**Proof.** First notice that $\bar{\phi} \in \Gamma$, since $\bar{\phi}$ is non-decreasing and $\bar{\phi}(-\infty) = 0$, $\bar{\phi}(+\infty) = k$, besides $\bar{\phi}(z) \to \frac{k(\eta-\mu_1)}{\lambda_1 - \mu_1 + \eta}$ when $z \to 0$ and $\bar{\phi}$ was built differently. second, let us prove the inequality (6). In effect, let $z < 0$ be, then

$$\begin{aligned} L_{\bar{\phi}} &\leq \bar{\phi}''(z) - c\bar{\phi}'(z) + p\bar{\phi}(z - c\tau) - d\bar{\phi}(z) \\ &= \frac{k(\eta - \mu_1)e^{z\lambda_1}}{\lambda_1 - \mu_1 + \eta}(\lambda_1^2 - c\lambda_1 - d + pe^{-\lambda_1 c\tau}) \\ &= \frac{k(\eta - \mu_1)e^{z\lambda_1}}{\lambda_1 - \mu_1 + \eta}\psi_0(\lambda_1, c) \\ &= 0. \end{aligned}$$

On the other hand, if $z \geq 0$ then we have

$$\begin{aligned} L_{\bar{\phi}} &\leq \bar{\phi}''(z) - c\bar{\phi}'(z) + \frac{pk}{1 + ak^q} - d\bar{\phi}(z) \\ &= -\frac{\lambda_1 k e^{z(\mu_1 - \eta)}}{\lambda_1 - \mu_1 + \eta}[(\mu_1 - \eta)^2 - c(\mu_1 - \eta)] + \frac{pk}{1 + ak^q} - d\left(k - \frac{\lambda_1 k}{\lambda_1 - \mu_1 + \eta} e^{z(\mu - \eta)}\right) \\ &= -\frac{\lambda_1 K e^{z(\mu_1 - \eta)}}{\lambda_1 - \mu_1 + \eta}[(\mu_1 - \eta)^2 - c(\mu_1 - \eta) - d] + \frac{pk}{1 + ak^q} - dk \\ &= -\frac{\lambda_1 k e^{z(\mu_1 - \eta)}}{\lambda_1 - \mu_1 + \eta}[\psi_1(\mu_1, c) + \eta^2 + \eta(c - 2\mu_1) - p_1 e^{-\mu_1 c\tau}] \\ &= -\frac{\lambda_1 k e^{z(\mu_1 - \eta)}}{\lambda_1 - \mu_1 + \eta}[\eta^2 + \eta(c - 2\mu_1) - p_1 e^{-\mu_1 c\tau}]. \end{aligned}$$

Note that $v(\eta) = \eta^2 + \eta(c - 2\mu_1) - p_1 e^{-\mu_1 c\tau} \geq 0$, for all $\eta \geq \eta_0 > 0$, since $\eta_0$ is root for $v(\eta)$. Hence $L_{\bar{\phi}} \leq 0$.

Now, for us to build a lower solution of (7), we will follow the ideas of Wu and Zou in [1]. Chose $\varepsilon > 0$ such that $\varepsilon < \lambda_1 < \lambda_1 + \varepsilon < \lambda_2$. Then we provide the following proposition.

**Proposition 4.** For any $c > c_*$ and $M > 1$ The function

$$\underline{\phi}(z) = \begin{cases} \dfrac{k(\eta - \mu_1)}{\lambda_1 - \mu_1 + \eta}(1 - Me^{\varepsilon z})e^{\lambda_1 z} & \text{si } z < \frac{1}{\varepsilon}\ln\left(\frac{1}{M}\right) \\ 0 & \text{si } z \geq \frac{1}{\varepsilon}\ln\left(\frac{1}{M}\right), \end{cases} \tag{15}$$

where $M > -\dfrac{apk_0 e^{-\lambda_1 c\tau(q+1)}(1 + e^{-\varepsilon c\tau})^{q+1}}{\psi_0(\lambda_1 + \varepsilon, c)}$, with $k_0 = \frac{k(\eta-\mu_1)}{\lambda_1 - \mu_1 + \eta}$; is lower solution of (7).

**Proof.** For $z \geq \frac{1}{\varepsilon}\ln\left(\frac{1}{M}\right)$ is easy to see that $L_{\underline{\phi}} \geq 0$. Now let $z < \frac{1}{\varepsilon}\ln\left(\frac{1}{M}\right)$ be, and we note that $\frac{1}{1+a[\phi_z(-c\tau)]^q} \geq 1 - a\phi_z^q(-c\tau)$ for $q > 0$. Then

$$\begin{aligned} L_{\underline{\phi}} \geq& \underline{\phi}''(z) - c\underline{\phi}'(z) + p\underline{\phi}(z - c\tau)[1 - a\underline{\phi}^q(z - c\tau)] - d\underline{\phi}(z) \\ =& k_0 e^{z\lambda_1}[\lambda_1^2 - M(\lambda_1 + \varepsilon)^2 e^{z\varepsilon}] - ce^{z\lambda_1}[\lambda_1 - M(\lambda_1 + \varepsilon)e^{z\varepsilon}] \\ &+ pe^{(z-c\tau)\lambda_1}[1 - Me^{(z-c\tau)\varepsilon}][1 - ak_0(e^{(z-c\tau)\lambda_1}(1 - Me^{(z-c\tau)\varepsilon}))^q] - de^{z\lambda_1}(1 - Me^{z\varepsilon}) \\ =& k_0 e^{z\lambda_1}[(\lambda_1^2 - c\lambda_1 - d + pe^{-\lambda_1 c\tau}) - Me^{z\varepsilon}((\lambda_1 + \varepsilon)^2 - c(\lambda_1 + \varepsilon) + pe^{-(\lambda_1+\varepsilon)c\tau}) \\ &- ak_0 pe^{z\lambda_1 q}e^{-\lambda_1 c\tau(q+1)}(1 - Me^{\varepsilon(z-c\tau)})^{q+1})] \end{aligned}$$

furthermore $(1 - Me^{(z-c\tau)\varepsilon})^{q+1} < (1 + e^{-\varepsilon c\tau})^{q+1}$, since $1 - Me^{(z-c\tau)\varepsilon} > 0$. Thus

$$\begin{aligned} L_{\underline{\phi}} \geq& k_0 e^{z\lambda_1}[(\lambda_1^2 - c\lambda_1 - d + pe^{-\lambda_1 c\tau}) - Me^{z\varepsilon}((\lambda_1 + \varepsilon)^2 - c(\lambda_1 + \varepsilon) - d + pe^{-(\lambda_1+\varepsilon)c\tau}) \\ &- ak_0 pe^{z\lambda_1 q}e^{-\lambda_1 c\tau(q+1)}(1 + e^{-\varepsilon c\tau})^{q+1}] \\ =& k_0 e^{z\lambda_1}[\psi_0(\lambda_1, c) - Me^{z\varepsilon}\psi_0(\lambda_1 + \varepsilon, c) - ak_0 pe^{z\lambda_1 q}e^{-\lambda_1 c\tau(q+1)}(1 + e^{-\varepsilon c\tau})^{q+1}] \end{aligned}$$

and choosing $0 < \varepsilon < \lambda_1 q$, we have

$$\begin{aligned} L_{\underline{\phi}} \geq& k_0 e^{z(\lambda_1+\varepsilon)}\left[-M\psi_0(\lambda_1 + \varepsilon, c) - ak_0 pe^{-\lambda_1 c\tau(q+1)}(1 + e^{-\varepsilon c\tau})^{q+1}\right] \\ =& k_0 e^{z(\lambda_1+\varepsilon)}(-\psi_0(\lambda_1 + \varepsilon, c))\left[M + \frac{ak_0 pe^{-\lambda_1 c\tau(q+1)}(1 + e^{-\varepsilon c\tau})^{q+1}}{\psi_0(\lambda_1 + \varepsilon, c)}\right]. \end{aligned}$$

We note that $\psi_0(\lambda_1 + \varepsilon, c) < 0$. Hence $L_{\underline{\phi}} \geq 0$.

In order to apply Theorem 2, first note that the condition **(S1)** is immediate by the definition of $\underline{\phi}$ and it is also easy to verify the condition **(S2)**. Then, for the case $c > c_*$, we obtain our main result on the existence of monotone traveling wave fronts for equation (1).

# 4. Existence of traveling wavefront solution: Case $c = c_*$

In this section we will prove the existence of a traveling wavefront at minimum speed $c = c_*$. In order to do that, we construct a lower and a upper solution to (7).

**Remark 1.** there exist a minimum speed $c_* = c_*(\tau, p, d) > 0$ and a corresponding number $\lambda_* = \lambda(c_*) > 0$ satisfying the following equation

$$p\tau e^{\frac{2 - c^2\tau - \sqrt{(c^2\tau - 2)^2 + 4c^2\tau(d\tau + 1)}}{2}} + 1 = \frac{c^2\tau - 2 + \sqrt{(c^2\tau - 2)^2 + 4c^2\tau(d\tau + 1)}}{c^2\tau}.$$

We notice that $(\lambda_*, c_*)$ is solution the system

$$\begin{aligned} h_1(\lambda, c) &= h_2(\lambda, c) \\ h_1'(\lambda, c) &= h_2'(\lambda, c), \end{aligned}$$

where $h_1(\lambda, c) = pe^{-\lambda c\tau}$, $h_2(\lambda, c) = d + c\lambda - \lambda^2$, since and $(\lambda_*, c_*)$ is the tangent point of $h_1$ and $h_2$ and $\frac{\partial \psi_0}{\partial c} < 0$. So the equation (12) has a double root $\lambda_*$.

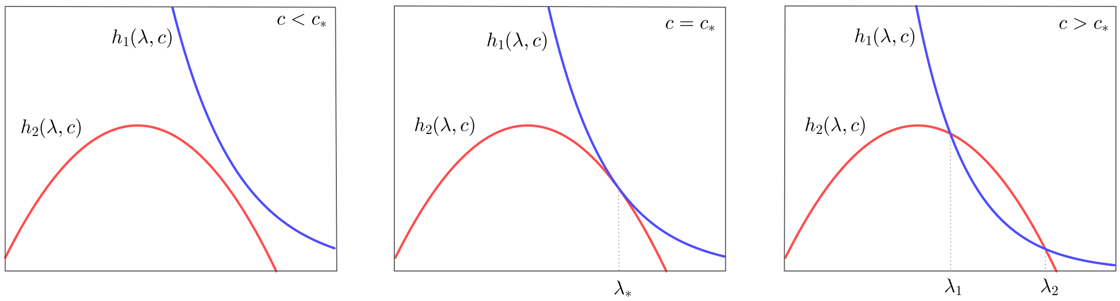



Figura 1: The graphs of $h_1$ and $h_2$ for $c < c_*$, $c = c_*$ and $c > c_*$, respectively.

**Proposition 5.** For any $c = c_*$ and $\eta_* \geq \eta_0 > 0$. The function

$$\bar{\phi}_*(z) = \begin{cases} \dfrac{k(\eta_* - \mu_1)}{\lambda_* + 2(\eta_* - \mu_1)}(2 - \lambda_* z)e^{z\lambda_*} & \text{si } z < 0 \\ k - \dfrac{\lambda_* k}{\lambda_* + 2(\eta_* - \mu_1)} e^{z(\mu_1 - \eta_*)} & \text{si } z \geq 0, \end{cases} \tag{16}$$

where $\eta_0$ is the same as that given in the proposition 3, is upper solution of (7).

**Proof.** First notice that $\bar{\phi}_* \in \Gamma$, since $\bar{\phi}_*$ is non-decreasing and $\bar{\phi}(-\infty) = 0$, $\bar{\phi}_*(+\infty) = k$, besides $\bar{\phi}_*(z) \to \frac{2k(\eta_* - \mu_1)}{\lambda_* + 2(\eta_* - \mu_1)}$ when $z \to 0$ and $\bar{\phi}_*$ was built differently. second, let us prove the inequality (6). In effect, let $z < 0$ be, then

$$\begin{aligned}
L_{\bar{\phi}_*} &\leq \bar{\phi}''_*(z) - c\bar{\phi}'_*(z) + p\bar{\phi}_*(z - c\tau) - d\bar{\phi}_*(z) \\
&= \frac{k(\eta_* - \mu_1)}{\lambda_* + 2(\eta_* - \mu_1)}[-\lambda_*^3 z e^{\lambda_* z} - c_*(\lambda_* e^{\lambda_* z} - \lambda_*^2 z e^{\lambda_* z}) + p\left(2 - \lambda_*(z - c_*\tau)\right) e^{\lambda_*(z - c_*\tau)} \\
&\quad - d(2 - \lambda_* z)e^{\lambda_* z}] \\
&= \frac{k(\eta_* - \mu_1)e^{\lambda_* z}}{\lambda_* + 2(\eta_* - \mu_1)}\left[-\lambda_*^3 z - c\lambda_* + c_*\lambda_*^2 z + p\left(2 - \lambda_*(z - c_*\tau)\right) e^{-\lambda_* c_*\tau} - d(2 - \lambda_* z)\right] \\
&= \frac{k(\eta_* - \mu_1)e^{\lambda_* z}}{\lambda_* + 2(\eta_* - \mu_1)}\left[-\lambda_* z(\lambda_*^2 - c\lambda - d + pe^{-\lambda_* c\tau}) - c\lambda_* - 2d + 2pe^{-\lambda_* c\tau} + p\lambda_* c_*\tau e^{-\lambda_* c_*\tau}\right] \\
&= \frac{k(\eta_* - \mu_1)e^{\lambda_* z}}{\lambda_* + 2(\eta_* - \mu_1)}\left[(2 - \lambda_* z)(\lambda_*^2 - c\lambda - d + pe^{-\lambda_* c\tau}) - \lambda_*(2\lambda_* - c_* - pc_*\tau e^{-\lambda_* c_*\tau})\right] \\
&= \frac{k(\eta_* - \mu_1)e^{\lambda_* z}}{\lambda_* + 2(\eta_* - \mu_1)}\left[(2 - \lambda_* z)\psi_0(\lambda_*, c_*) - \lambda_*\psi'_0(\lambda_*, c_*)\right]
\end{aligned}$$

Observe that $\lambda_*$ is double root of (12), then $\psi_0(\lambda_*, c_*) = \psi'_0(\lambda_*, c_*) = 0$, therefore $L_{\bar{\phi}} \leq 0$. On the other hand, if $z \geq 0$, then we have similarly how it was done in the proposition 3, that $L_{\bar{\phi}} \leq 0$.

**Remark 2.** In proposition 4, the construction of the lower solution for (7) depends on the existence of some $\varepsilon > 0$, such that $\varepsilon < \lambda_1 < \lambda_1 + \varepsilon < \lambda_2$, where $\lambda_1$ and $\lambda_2$ are the two positive real roots of Equation (12), however such construction does not apply to the case where $\lambda_1 = \lambda_2 = \lambda_*$ is a double root.

**Proposition 6.** For any $c = c_*$ and $0 < b < \lambda_* q$. There exists $N > 0$ such that the function

$$\underline{\phi}_*(z) = \begin{cases} N(b^{-1}e^{bz-1} - z)e^{\lambda_* z} & \text{si } z < b^{-1} \\ 0 & \text{si } z \geq b^{-1}, \end{cases} \tag{17}$$

is lower solution of (7).

**Proof.** First notice that $\underline{\phi}_*$ was built differently and $\underline{\phi}_*(z) \to 0$ when $z \to b^{-1}$ . Second, the differential inequality for the case $z \geq b^{-1}$ is easily obtained. Let us prove it in the case $z < b^{-1}$. Then we have

$$\begin{aligned}
L_{\underline{\phi}_*} \geq & \underline{\phi}_*''(z) - c\underline{\phi}_*'(z) + p\underline{\phi}_*(z - c\tau)[1 - a\underline{\phi}_*^q(z - c\tau)] - d\underline{\phi}_*(z) \\
=& Ne^{\lambda_* z}[(b^{-1}e^{bz-1}(\lambda_* + b)^2 - 2\lambda_* - \lambda_*^2 z) - c(b^{-1}e^{bz-1}(\lambda_* + b) - z\lambda_* - 1) \\
& + pe^{-\lambda_* c\tau}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)(1 - aN^q e^{\lambda_* q(z-c\tau)}(b^{-1}e^{b(z-c\tau-)1} - z + c\tau)^q) \\
& - d(b^{-1}e^{bz-1} - z)] \\
=& Ne^{\lambda_* z}[b^{-1}e^{bz-1}((\lambda_* + b)^2 - c(\lambda_* + b) - d) - z(\lambda_*^2 - c\lambda_* - d) - 2\lambda_* + c \\
& + pe^{-\lambda_* c\tau}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)(1 - aN^q e^{\lambda_* q(z-c\tau)}(b^{-1}e^{b(z-c\tau-)1} - z + c\tau)^q)] \\
=& Ne^{\lambda_* z}[b^{-1}e^{bz-1}((\lambda_* + b)^2 - c(\lambda_* + b) - d) - z(\lambda_*^2 - c\lambda_* - d) - 2\lambda_* + c \\
& + pe^{-\lambda_* c\tau}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)(1 - aN^q e^{\lambda_* q(z-c\tau)}(b^{-1}e^{b(z-c\tau-)1} - z + c\tau)^q)] \\
=& Ne^{\lambda_* z}[b^{-1}e^{bz-1}((\lambda_* + b)^2 - c(\lambda_* + b) - d) - z(\lambda_*^2 - c\lambda_* - d) - 2\lambda_* + c \\
& + pe^{-\lambda_* c\tau}b^{-1}e^{b(z-c\tau)-1} - pe^{-\lambda_* c\tau}z + pe^{-\lambda_* c\tau}c\tau \\
& - paN^q e^{-\lambda_* c\tau}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)^{q+1}e^{\lambda_* q(z-c\tau)}] \\
=& Ne^{\lambda_* z}[b^{-1}e^{bz-1}((\lambda_* + b)^2 - c(\lambda_* + b) - d + pe^{-c\tau(\lambda_* + b)}) - z(\lambda_*^2 - c\lambda_* - d + pe^{-\lambda_* c\tau}) \\
& - (2\lambda_* - c - pe^{-\lambda_* c\tau}c\tau) - paN^q e^{-\lambda_* c\tau}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)^{q+1}e^{\lambda_* q(z-c\tau)}] \\
=& Ne^{\lambda_* z}[b^{-1}e^{bz-1}\psi_0(\lambda_* + b, c_*) - z\psi_0(\lambda_*, c_*) - \psi_0'(\lambda_*, c_*) \\
& - paN^q e^{-\lambda_* c\tau(q+1)}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)^{q+1}e^{\lambda_* qz}] \\
=& Ne^{\lambda_* z}[b^{-1}e^{bz-1}\psi_0(\lambda_* + b, c_*) - paN^q e^{-\lambda_* c\tau(q+1)}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)^{q+1}e^{\lambda_* qz}] \\
=& Ne^{(\lambda_* + b)z}\left[b^{-1}e^{-1}\psi_o(\lambda_* + b, c_*) - paN^q e^{-\lambda_* c\tau(q+1)}(b^{-1}e^{b(z-c\tau)-1} - z + c\tau)^{q+1}e^{(\lambda_* q - b)z}\right].
\end{aligned}$$

The function

$$\xi(z) = (b^{-1}e^{b(z-c\tau)-1} - z + c\tau)^{q+1}e^{(\lambda_* q - b)z}, \quad q > 0$$

satisfies $\xi > 0$ for all $z < b^{-1}$ and $\lim_{z \to -\infty} \xi(z) = 0$ because $b < \lambda_* q$. Let $\xi_* = \max_{z < b^{-1}} \xi(z)$. Then we have

$$L_{\underline{\phi}_*} \geq Ne^{(\lambda_* + b)z}\left[b^{-1}e^{-1}\psi_o(\lambda_* + b, c_*) - paN^q e^{-\lambda_* c\tau(q+1)}\xi_*\right] \tag{18}$$

and $L_{\underline{\phi}_*} \geq 0$ if we choose $N^q \leq \frac{\psi_0(\lambda_* + b, c_*)}{bpa\xi_* e^{1 - \lambda_* c\tau(q+1)}}$, from which concludes the proof.

**lemma 3.** $\underline{\phi}_*(z)$ and $\bar{\phi}_*(z)$, satisfy the condition **(S2)**.

**Proof.** Note that

$$\lim_{z \to -\infty} \frac{\bar{\phi}_*(z)}{\underline{\phi}_*(z)} = \lim_{z \to -\infty} \frac{\frac{K(\eta_* - \mu_1)(2 - \lambda_* z)}{\lambda_* + 2(\eta_* - \mu_1)}e^{z\lambda_*}}{N(b^{-1}e^{bz-1} - z)e^{\lambda_* z}} = \frac{\frac{K(\eta_* - \mu_1)\lambda_*}{\lambda_* + 2(\eta_* - \mu_1)}}{N}.$$

Then, if we choose $N < \frac{K(\eta_*-\mu_1)\lambda_*}{\lambda_*+2(\eta_*-\mu_1)}$ , there exist $T \gg 1$ such that $\underline{\phi}_*(z) < \bar{\phi}_*(z)$ for all $z \in (-\infty, -T]$. it is easy to prove that $\bar{\phi}_*$ is an increasing function this implies that $\min_{z\in[-T,b^{-1}]} \bar{\phi}_*(z) = \bar{\phi}_*(-T)$.

Let $m_0 = \max_{z\in[-T,b^{-1}]} (b^{-1}e^{bz-1} - z)e^{\lambda_* z}$. If we choose $N$ satisfying $Nm_0 < \bar{\phi}_*(-T)$, then then we have

$$\underline{\phi}_*(z) < Nm_0 < \bar{\phi}_*(-T) < \bar{\phi}_*(z).$$

Therefore, if

$$N \leq \min\left\{\frac{k(\eta_* - \mu_1)\lambda_*}{\lambda_* + 2(\eta_* - \mu_1)}, \frac{\bar{\phi}_*(-T)}{m_0}\right\}$$

hypothesis **(S2)** is satisfied.

In order to apply Theorem 2, we note that the condition **(S1)** is immediate by the definition of $\underline{\phi}_*$. Then, for the case $c = c_*$, we obtain our main result on the existence of monotone traveling wave fronts for equation (1).

## 5. Numerical Simulations

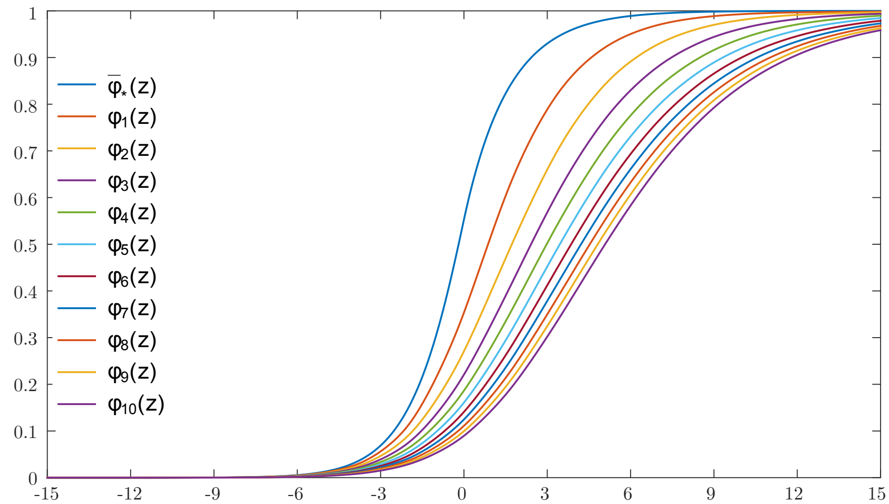
In this section, we present some numerical simulations. We find that these numerical results confirm our theoretical results shown in section 3 and section 4. The computational results reported in this section, to numerically approximate the traveling wavefront, are based on the following monotonous iteration scheme that arise from the results given in [1].

$$\begin{cases} \phi_{n+1}(z) = \dfrac{p}{d(\alpha_2 - \alpha_1)} \left[\displaystyle\int_{-\infty}^{z} e^{\alpha_1(z-s)} H(\phi_n(s))ds + \int_{z}^{+\infty} e^{\alpha_2(z-s)} H(\phi_n(s))ds\right] \\ \phi_0(z) = \bar{\phi}(z) \quad (\text{or} \quad \bar{\phi}_*(z)), \end{cases} \tag{19}$$

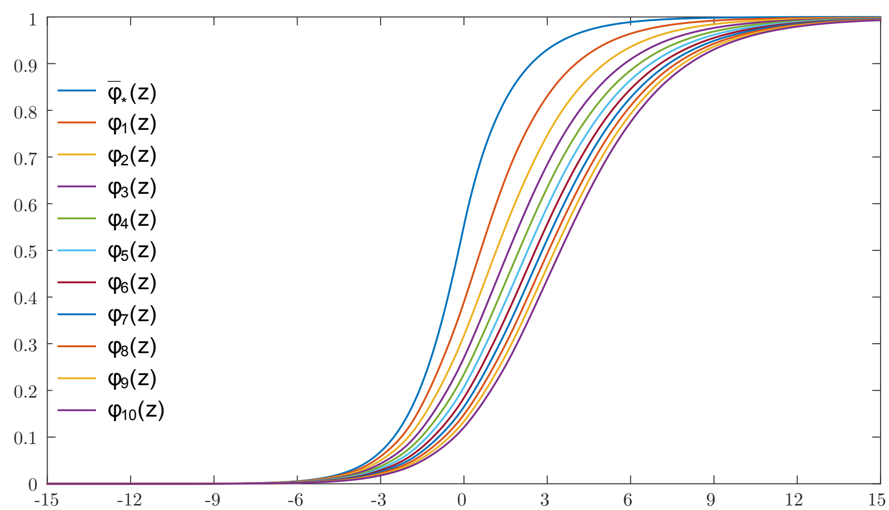
where

$$H(\phi_n(z)) = \frac{\phi_n(z - cr)}{1 + a\phi_n^q(z - cr)},$$

and $\alpha_1 < 0 < \alpha_2$ are roots of $x^2 - cx - 1 = 0$. Now to approximate the traveling wavefront that moves at minimum speed $c = c_*$, we take some particular values for the parameters $a = e - 1$, $\tau = 1$, $p = e$ and $d = 1$ then $c_*(\tau, p, d) = 1$. The graphs of first 10 iterations for $q = 1$ and $q = 1{,}5$ respectively are plotted.



$\overline{\varphi}_*(z)$
$\varphi_1(z)$
$\varphi_2(z)$
$\varphi_3(z)$
$\varphi_4(z)$
$\varphi_5(z)$
$\varphi_6(z)$
$\varphi_7(z)$
$\varphi_8(z)$
$\varphi_9(z)$
$\varphi_{10}(z)$
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
-15 -12 -9 -6 -3 0 3 6 9 12 15

$\overline{\varphi}_*(z)$
$\varphi_1(z)$
$\varphi_2(z)$
$\varphi_3(z)$
$\varphi_4(z)$
$\varphi_5(z)$
$\varphi_6(z)$
$\varphi_7(z)$
$\varphi_8(z)$
$\varphi_9(z)$
$\varphi_{10}(z)$
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
-15 -12 -9 -6 -3 0 3 6 9 12 15

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